

MAS317 Linear Algebra II, 2007

Solutions to Coursework 6.

1. (a). $(AA^T)^T = (A^T)^T A^T = AA^T$, so AA^T is symmetric. [5 marks]

(b). $(A + B)^T = A^T + B^T = A + B$, so $A + B$ is symmetric. [5 marks]

(c). $(AB)^T = B^T A^T = BA$ not AB , so AB probably not symmetric.

Example of two symmetric matrices whose product is not

symmetric: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$. [10 marks]

2. (a) A, B, C, D, E are all symmetric and F is not symmetric, so F is not congruent to any of A, B, C, D, E over $\mathbb{C}$. [2 marks]

A and E have row rank 1, and B and C and D all have row rank 2.

This should be obvious to you, but if not then reduce to echelon form and see how many rows are non-zero.

So A and E are congruent over $\mathbb{C}$, and B and C and D are all congruent over $\mathbb{C}$. [8 marks]

(b) For normal forms over $\mathbb{R}$, use row and corresponding column operations. [5 × 5 marks]

$A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \xrightarrow[\substack{R_2 \rightarrow R_2 - R_1 \\ C_2 \rightarrow C_2 - C_1}]{\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}}$ So normal form over $\mathbb{R}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \xrightarrow[\substack{R_1 \rightarrow R_1 + R_2 \\ C_1 \rightarrow C_1 + C_2}]{\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}} \xrightarrow[\substack{R_2 \rightarrow R_2 - 1/2 R_1 \\ C_2 \rightarrow C_2 - 1/2 C_1}]{\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}}$ So normal form over $\mathbb{R}$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$C = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 \rightarrow R_2 - 2/3 R_1 \\ C_2 \rightarrow C_2 - 2/3 C_1}]{\begin{pmatrix} 3 & 0 \\ 0 & -1/3 \end{pmatrix}}$ So normal form over $\mathbb{R}$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$D = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \xrightarrow[\substack{R_2 \rightarrow R_2 + R_1 \\ C_2 \rightarrow C_2 + C_1}]{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ and this is the normal form over $\mathbb{R}$.

$E = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xrightarrow[\substack{R_2 \rightarrow R_2 + R_1 \\ C_2 \rightarrow C_2 + C_1}]{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}}$ and this is the normal form over $\mathbb{R}$

(c) Now B and C are the only pair with the same normal form over $\mathbb{R}$, and so are the only pair that are congruent under $\mathbb{R}$. [5 marks]

$$3. A = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 5 & -1 \\ 2 & -1 & -4 \end{pmatrix}$$

$$R_2 \leftrightarrow R_1 \begin{pmatrix} 3 & 5 & -1 \\ 0 & 3 & 2 \\ 2 & -1 & -4 \end{pmatrix} \quad C_2 \leftrightarrow C_1 \begin{pmatrix} 5 & 3 & -1 \\ 3 & 0 & 2 \\ -1 & 2 & -4 \end{pmatrix}$$

$$\begin{matrix} R_2 \rightarrow R_2 - (3/5)R_1 \\ R_3 \rightarrow R_3 + (1/5)R_1 \end{matrix} \begin{pmatrix} 5 & 3 & -1 \\ 0 & -9/5 & 13/5 \\ 0 & 13/5 & -21/5 \end{pmatrix} \begin{matrix} C_2 \rightarrow C_2 - (3/5)C_1 \\ C_3 \rightarrow C_3 + (1/5)C_1 \end{matrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & -9/5 & 13/5 \\ 0 & 13/5 & -21/5 \end{pmatrix}$$

$$R_3 \rightarrow R_3 + \frac{13}{9}R_2 \begin{pmatrix} 5 & 0 & 0 \\ 0 & -9/5 & 13/5 \\ 0 & 0 & -4/9 \end{pmatrix}$$

$$C_3 \rightarrow C_3 + \frac{13}{9}C_2 \begin{pmatrix} 5 & 0 & 0 \\ 0 & -9/5 & 0 \\ 0 & 0 & -4/9 \end{pmatrix} = D. \quad [20 \text{ marks}]$$

To find P start with I_3 and just do the col operations above.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C_2 \leftrightarrow C_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} C_2 \rightarrow C_2 - (3/5)C_1 \\ C_3 \rightarrow C_3 + (1/5)C_1 \end{matrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3/5 & 1/5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_3 \rightarrow C_3 + \frac{13}{9}C_2 \begin{pmatrix} 0 & 1 & 13/9 \\ 1 & -3/5 & -2/3 \\ 0 & 0 & 1 \end{pmatrix} = P. \quad [10 \text{ marks}]$$

To find the rank and signature of A as a matrix over $\mathbb{R}$: there are 1 positive and 2 negative entries in D , so $p = 1$ and $q = 2$ and

so the rank is $3(= 1 + 2)$ and the signature is $-1(= 1 - 2)$. **[6 marks]**

Normal form over $\mathbb{R}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and over $\mathbb{C}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

[4 marks]